Scale-Free Network without a Power-Law Degree Distribution

Xiangyi Meng[1] and Bin Zhou[2]
[1] *Network Science Institute and Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA*
[2] *School of Management Science and Engineering, Nanjing University of Information Science and Technology, Nanjing, 210044, China*

Keywords: Scale-free network, Degree–degree distance, Preferential attachment, Fitness

Originating from the ideas of renormalization group in statistical field theories [1], the study of the *scale-free* property in network theories has undergone an impressive development over the past twenty years [2, 3]. It is noteworthy that the term "scale-free" [2] was first dubbed to denote the unique property that a power-law distribution of, e.g., degree k, $f(k) \propto k^{-\alpha}$ is invariant (free) under the continuous scale transformation $k \to k + \epsilon k$. However, since a pure power-law degree distribution (DD) would not be normalizable in the domain $k \in [0, \infty)$, the DD cannot truly be power law but requires an ultraviolet (UV) cutoff, in terms of either a $k_{\min}$ (the minimum degree each node can have) or some other nontrivial corrections around small k. The DD regains its scale invariance only asymptotically in the infrared (IR) limit $k \to \infty$.

Therefore, a finite-size network can only be approximately "scale-free" [4]. This has put great difficulty in how we can test if real-world finite-size complex networks are "scale-free" in *abundance*, a remarkable claim that, albeit supported by many empirical observations [5], remains controversial in recent literature [6]. Without prior knowledge of the UV cutoff, it is unknown how large a typical degree k must be, that we should consider as already entering the power-law regime where rigorous statistical analysis can be employed. An oversimplified solution to this is to statistically test the full domain of k, ignoring possible UV cutoffs, resulting in that less than one third of real-world networks have a statistically significant power-law DD [7]. Yet, when another metric is investigated, namely the degree–degree distance η [7], defined by

$$\eta = \exp |\ln k_i - \ln k_j| \tag{1}$$

for every link $i \leftrightarrow j$ connecting two nodes i and j, it turns out that real-world networks almost universally have a statistically significant power-law degree–degree distance distribution (DDDD) [7]. Much interest has since been drawn to the characteristics of η, bringing its prevalence to broader network science topics [8], especially network closeness [9] and network assortativity [10].

Despite the potentials, the finding of power-law DDDD raises a question: *is there any theoretical relevance between the power laws (if any) of DD and DDDD?* It appears that asymptotically, a power law of DDDD $g(\eta) \propto \eta^{-\beta}$ is nothing but a delegate of the power law of DD of the same network, given that an equality $\beta = \alpha - 1$ [7] has been derived for both the Barabási–Albert (BA) model [2] and a special power-law-distributed fitness model [7]. There seems no reason to investigate the power law of DDDD for its own theoretical purpose.

Nevertheless, here we show that the power law of DDDD is *more than* a delegate. Our main result is that the set of networks with an asymptotic power-law DD is a *proper subset* of those with an asymptotic power-law DDDD. This immediately indicates that there are networks whose DD is not power law, but DDDD is, differing not only in statistical significance but also in their asymptotic limits. This also implies that our current understanding of the scale-free property of networks in terms of only the power law of DD is incomplete. Indeed, given the broader scope of power-law DDDD, we propose that it would be more appropraite and general to denote "scale-free networks" as having a power-law distribution for *any* of its metrics, not the degree only. Such, the scale-free property need not manifest in all metrics, thus better identified and distinguished not by apparent power laws but by the underlying network mechanisms, such as *preferential attachment* [2] or *quenched fitness* [11]. In particular, we will show that either of these two fundamental mechanisms can generate networks as concrete examples for our purpose—representing scale-free networks without a power-law DD.

Results.—We (abusively) phrase our main result as

$$\mathcal{D}^2|_{\text{power-law}} \subset \mathcal{D}^4|_{\text{power-law}} \tag{2}$$

and derive it in two steps, claiming its (i) *inclusion* and, more interestingly, its (ii) *strict inequality* as follows:

(i) $\mathcal{D}^2|_{\text{power-law}} \subseteq \mathcal{D}^4|_{\text{power-law}}$, i.e., *every network with a power-law DD also has a power-law DDDD.*

(ii) $\left|\mathcal{D}^2|_{\text{power-law}}\right| < \left|\mathcal{D}^4|_{\text{power-law}}\right|$, i.e. *there are networks that do not have a power-law DD but exhibit a power-law DDDD.* We will consider two network models of general interest:

Preferential attachment of internal links only.—As our first model, this "no-growth" model differs from the BA model [2] in that the number of nodes N of the network is fixed as a constant, and only internal links are added to the initially empty network during its evolution. At each time step t, two nodes i and j are randomly and independently chosen, resulting in a link drawn between

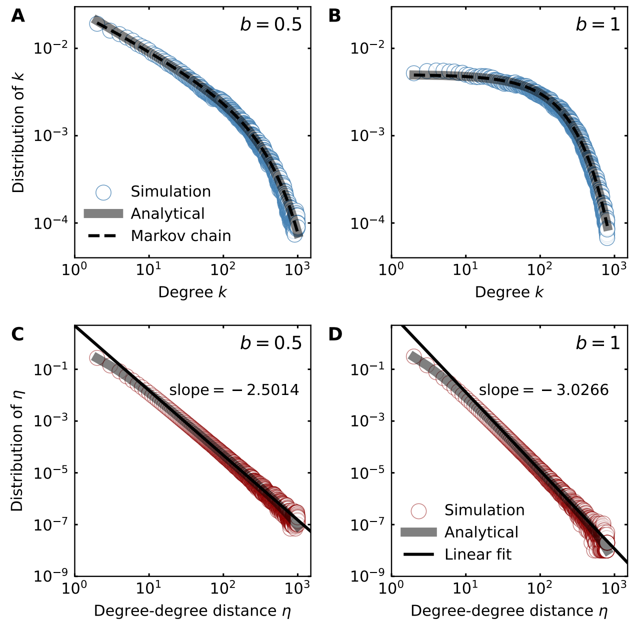

FIG. 1: Distributions of (a-b) k and (c-d) η of the "no-growth" model, generated by preferential attachment of internal links only (with attachment probability $\propto k+$ a small constant b). Simulation results (circle): average of 10^2 runs on $N=10^4$ nodes and $T=10^6$ links.

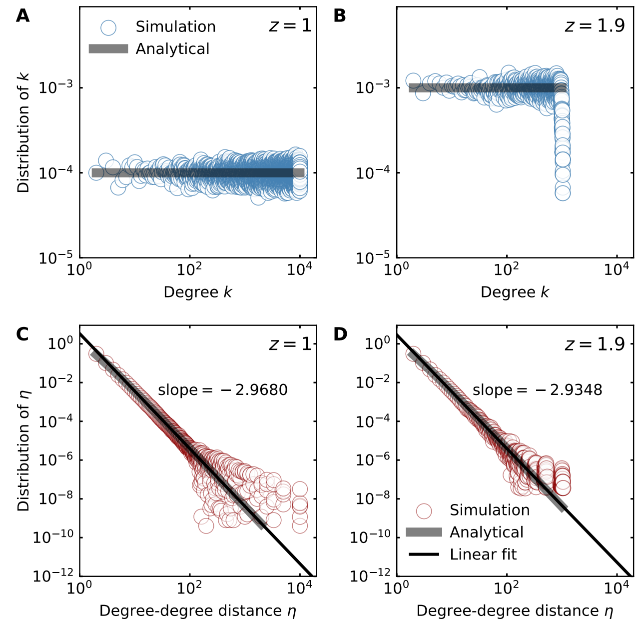

FIG. 3: Distributions of (a-b) k and (c-d) η of the "strong-coupling" model, a uniform-distributed fitness model (fitness $\omega \in [0, \omega_{\max}=1]$) where two nodes are linked if their fitness sum is larger than a threshold z. Simulation results (circle): average of 10^2 runs on $N=10^4$ nodes.

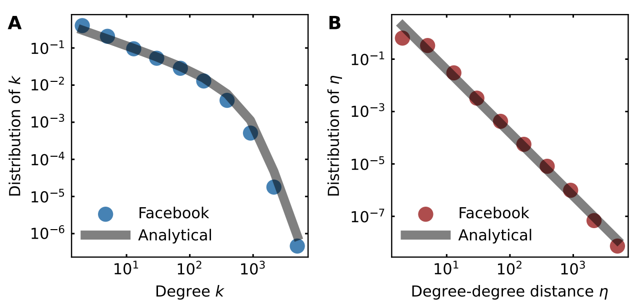

FIG. 2: Distributions of (a) k and (b) η of the Facebook network, fitted by analytical results (solid line) of the "no-growth" model with $b=0.4092$.

i and j. The probability of choosing such two nodes is $\propto (k_i+b)(k_j+b)$, i.e., preferentially proportional to each node's current degree k plus a small constant b. After T time steps, the network acquires T links.

We derive the analytical results for both DD and DDDD of the no-growth model. In contrast to DD, we find that the DDDD exhibits a strong power law in the small b regime. The analytical result matches the simulation result [Fig. 1(c-d)], with the power-law exponent of DDDD equal to $2+b$ as expected.

It is worth noting that as $b \to 0$, we rediscover the classic scaling $\sim \eta^{-2}$ for the DDDD of the BA model [7]. This is evidence that the scale-free property of the BA model indeed originates from the preferential attachment mechanism but *not* from network growth, a discrimination that cannot be revealed by comparing DD only [12].

Facebook is the world's largest social networking platform. Figure 2 shows the DD and DDDD of Facebook. We find that the DD of Facebook is not a power-law distribution, but DDDD is. Notably, both DD and DDDD are also in good agreement with the analytical results of the no-growth model, suggesting that the model is more than a toy model but of strong practical significance as well.

Fitness with threshold.—The second model of interest is defined by assigning a random fitness ω that follows a fitness distribution $\rho(\omega)$ to each node of the network [13]. For every two nodes i and j, let a link be drawn with probability $\sigma(\omega_i, \omega_j)$ that depends only on the fitness of the nodes, not their degrees [11]. By choosing $\sigma(\omega_i, \omega_j) = \theta(\omega_i + \omega_j - z)$, where $\theta(x)$ is the Heaviside step function, a strong coupling of fitness is introduced, such that a link will be deterministically drawn if and only if the sum of the fitnesses of the two nodes is larger than a threshold z. In particular, here we consider a uniform distribution $\rho(\omega)$ for $\omega \in [0, \omega_{\max}]$ and assume that $\omega_{\max} \le z \le 2\omega_{\max}$. We find that the annealed average DD is simply given by $f(k) \sim N^{-1}$ [Fig. 3(a-b)] following the calculation in Ref. [14]. For DDDD, we derive $g(\eta) \sim \eta^{-3}$ for large η, a strong power law that is independent of z [Fig. 3(c-d)].

Discussion.—We also observe that Eq. (2) establishes an inclusive order between the power laws of DD and DDDD. This raises the question of whether we can find another more inclusive metric beyond DDDD. It would be interesting if a hierarchy between all such metrics could be established, especially for scale-free networks, that offers new insights on distinguishing the origins of scale-free properties.

[1] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory: Volume 1, From Brownian Motion to Renormalization and Lattice Gauge Theory*, 1st ed. (Cambridge University Press, New York, 1989); *Statistical Field Theory: Volume 2, Strong Coupling, Monte Carlo Methods, Conformal Field Theory and Random Systems*, 1st ed. (Cambridge University Press, New York, 1989).

[2] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[3] L. A. Adamic and B. A. Huberman, Science **287**, 2115 (2000); P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000); P. L. Krapivsky, G. J. Rodgers, and S. Redner, **86**, 5401 (2001); C. Song, S. Havlin, and H. A. Makse, Nature **433**, 392 (2005); T. S. Evans and J. Saramäki, Phys. Rev. E **72**, 026138 (2005); S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008); F. Radicchi, J. J. Ramasco, A. Barrat, and S. Fortunato, Phys. Rev. Lett. **101**, 148701 (2008); M. Szell, R. Lambiotte, and S. Thurner, Proc. Natl. Acad. Sci. **107**, 13636 (2010); R. Lambiotte, M. Rosvall, and I. Scholtes, Nat. Phys. **15**, 313 (2019); T. Nesti, F. Sloothaak, and B. Zwart, Phys. Rev. Lett. **125**, 058301 (2020).

[4] M. Serafino, G. Cimini, A. Maritan, A. Rinaldo, S. Suweis, J. R. Banavar, and G. Caldarelli, Proc. Natl. Acad. Sci. **118**, e2013825118 (2021).

[5] A.-L. Barabási and E. Bonabeau, Sci. Am. **288**, 60 (2003); K.-I. Goh, E. Oh, H. Jeong, B. Kahng, and D. Kim, Proc. Natl. Acad. Sci. **99**, 12583 (2002); R. Pastor-Satorras, E. Smith, and R. V. Solé, J. Theor. Biol **222**, 199 (2003); R. Cohen, K. Erez, D. Ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000); K.-I. Goh, B. Kahng, and D. Kim, **87**, 278701 (2001).

[6] A. D. Broido and A. Clauset, Nat. Commun. **10**, 1017 (2019); P. Holme, **10**, 1016 (2019); R. E. Langendorf and M. G. Burgess, Sci. Rep. **11**, 20501 (2021); M. Serafino, G. Cimini, A. Maritan, A. Rinaldo, S. Suweis, J. R. Banavar, and G. Caldarelli, Proc. Natl. Acad. Sci. **118**, e2013825118 (2021).

[7] B. Zhou, X. Meng, and H. E. Stanley, Proc. Natl. Acad. Sci. **117**, 14812 (2020).

[8] B. Wang, J. Zhu, and D. Wei, Mod. Phys. Lett. B **35**, 2150331 (2021); A. J. Maren, Entropy **23**, 319 (2021).

[9] T. S. Evans and B. Chen, Commun. Phys. **5**, 1 (2022).

[10] A. Farzam, A. Samal, and J. Jost, Sci. Rep. **10**, 1 (2020).

[11] G. Caldarelli, A. Capocci, P. De Los Rios, and M. A. Muñoz, Phys. Rev. Lett. **89**, 258702 (2002).

[12] A.-L. Barabási, *Network Science*, 1st ed. (Cambridge University Press, Boston, 2016).

[13] G. Bianconi and A.-L. Barabási, EPL **54**, 436 (2001).

[14] V. D. P. Servedio, G. Caldarelli, and P. Buttà, Phys. Rev. E **70**, 056126 (2004).